New application of Dirac's representation: N-mode squeezing enhanced operator and squeezed state *

Xue-xiang Xu[1], Li-yun Hu[1,2†] and Hong-yi Fan[1]
[1]Department of Physics, Shanghai Jiao Tong University, Shanghai, 200240, China
[2]College of Physics & Communication Electronics, Jiangxi Normal University, Nanchang 330022, China

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Abstract

It is known that $\exp\left[\mathtt{i}\lambda\left(Q_1P_1-\mathtt{i}/2\right)\right]$ is a unitary single-mode squeezing operator, where Q_1,P_1 are the coordinate and momentum operators, respectively. In this paper we employ Dirac's coordinate representation to prove that the exponential operator $S_n\equiv\exp[\mathtt{i}\lambda\sum_{i=1}^{n}(Q_iP_{i+1}+Q_{i+1}P_i))]$, $(Q_{n+1}=Q_1,\ P_{n+1}=P_1)$, is a n-mode squeezing operator which enhances the standard squeezing. By virtue of the technique of integration within an ordered product of operators we derive S_n's normally ordered expansion and obtain new n-mode squeezed vacuum states, its Wigner function is calculated by using the Weyl ordering invariance under similar transformations.

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1 Introduction

Squeezed state has been a hot topic in quantum optics since Stoler [1] put forward the concept of the optical squeezing in 1970's. $S_1=\exp\left[\mathtt{i}\lambda\left(Q_1P_1-\mathtt{i}/2\right)\right]$ is a unitary single-mode squeezing operator, where Q_1, P_1 are the coordinate and momentum operators, respectively, λ is a squeezing parameter. Their variances in the squeezed state $S_1\left|0\right\rangle$ =sech$^{1/2}\lambda\exp\left[-\frac{1}{2}a_1^{\dagger 2}\tanh\lambda\right]\left|0\right\rangle$ are

$$\Delta Q_1=\frac{1}{4}e^{2\lambda},\ \Delta P_1=\frac{1}{4}e^{-2\lambda},\ (\Delta Q_1)(\Delta P_1)=\frac{1}{4}.$$

Some generalized squeezed state have been proposed since then. Among them the two-mode squeezed state not only exhibits squeezing, but also quantum entanglement between the idle-mode and the signal-mode in frequency domain, therefore is a typical entangled states of continuous variable. In recent years, various entangled states have attracted considerable attention and interests of physists because of their potential uses in quantum communication [2]. Theoretically, the two-mode squeezed state is constructed by acting the two-mode squeezing operator $S_2=\exp[\lambda(a_1a_2-a_1^{\dagger}a_2^{\dagger})]$ on the two-mode vacuum state $\left|00\right\rangle$[3, 4, 5],

$$S_2\left|00\right\rangle=\mathrm{sech}\lambda\exp\left[-a_1^{\dagger}a_2^{\dagger}\tanh\lambda\right]\left|00\right\rangle. \tag{1}$$

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†Corresponding author. E-mail addresses: hlyun2008@126.com.

We also have $S_2 = \exp\left[\mathtt{i}\lambda\left(Q_1P_2+Q_2P_1\right)\right]$, where Q_i and P_i are the coordinate and momentum operators related to Bose operators $(a_i, a_i^\dagger)$ by

$$Q_i = (a_i + a_i^\dagger)/\sqrt{2},\ P_i = (a_i - a_i^\dagger)/(\sqrt{2}\mathtt{i}) \tag{2}$$

In the state $S_2\left|00\right\rangle$, the variances of the two-mode quadrature operators of light field,

$$\mathfrak{X} = (Q_1+Q_2)/2,\ \mathfrak{P} = (P_1+P_2)/2,\quad [\mathfrak{X},\mathfrak{P}] = \frac{\mathtt{i}}{2}, \tag{3}$$

take the standard form, i.e.,

$$\left\langle 00\right| S_2^\dagger \mathfrak{X}^2 S_2 \left|00\right\rangle = \frac{1}{4}e^{-2\lambda},\quad \left\langle 00\right| S_2^\dagger \mathfrak{P}^2 S_2 \left|00\right\rangle = \frac{1}{4}e^{2\lambda},\ \text{ and } (\Delta\mathfrak{X})(\Delta\mathfrak{P}) = \frac{1}{4}. \tag{4}$$

On the other hand, the two-mode squeezing operator has a neat and natural representation in the entangled state $\left|\eta\right\rangle$ representation [6],

$$S_2 = \int \frac{d^2\eta}{\pi\mu}\left|\frac{\eta}{\mu}\right\rangle\left\langle \eta\right|, \tag{5}$$

where

$$\left|\eta\right\rangle = \exp(-\frac{1}{2}\left|\eta\right|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger)\left|00\right\rangle, \tag{6}$$

makes up a complete set

$$\int \frac{d^2\eta}{\pi}\left|\eta\right\rangle\left\langle \eta\right| = 1.$$

$\left|\eta\right\rangle$ was constructed according to the idea of quantum entanglement innitiated by Einstein, Podolsky and Rosen in their argument that quantum mechanics is incomplete [7].

An interesting question naturally arises: is the n-mode exponential operator

$$S_n \equiv \exp\left[\mathtt{i}\lambda\sum_{i=1}^{n}(Q_iP_{i+1}+Q_{i+1}P_i)\right],\quad (Q_{n+1}=Q_1,\ P_{n+1}=P_1),\ n \geqslant 2, \tag{7}$$

a squeezing operator? If yes, what kind of squeezing for n-mode quadratures of field it can engenders? To answer these questions we must know what is the normally ordered expansion of S_n and what is the state $S_n\left|\mathbf{0}\right\rangle$ ($\left|\mathbf{0}\right\rangle$ is the n-mode vacuum state)? In this work we shall analyse S_n in detail. But how to disentangle the exponential of S_n? Since the terms in the set Q_iP_{i+1} and $Q_{i+1}P_i$ $(i=1,2,\cdots,n)$ do not make up a closed Lie algebra, the problem of what is S_n's normally ordered form seems difficult. Thus we appeal to Dirac's coordinate representation and the technique of integration within an ordered product (IWOP) of operators [8, 9] to solve this problem. Our work is arranged as follows: firstly we use the IWOP technique to derive the normally ordered expansion of S_n and obtain the explicit form of $S_n\left|\mathbf{0}\right\rangle$; then we examine the variances of the n-mode quadrature operators in the state $S_n\left|\mathbf{0}\right\rangle$, we find that S_n causes squeezing which is stronger than the standard squeezing. Thus S_n is an n-mode squeezing-enhanced operator. The Wigner function of $S_n\left|\mathbf{0}\right\rangle$ is calculated by using the Weyl ordering invariance under similar transformations. Some examples are discussed in the last section.

2 Normal Product Form of S_n derived by Dirac's coordinate representation

In order to disentangle operator S_n, let A be

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{8}$$

then S_n in (7) is compactly expressed as

$$S_n=\exp[\mathtt{i}\lambda Q_i A_{ij} P_j], \tag{9}$$

here and henceforth the repeated indices represent Einstein's summation notation. Using the Baker-Hausdorff formula,

$$e^A B e^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\cdots,$$

we have

$$\begin{aligned} S_n^{-1}Q_k S_n &= Q_k-\lambda Q_i A_{ik}+\frac{1}{2!}\mathtt{i}\lambda^2[Q_i A_{ij}P_j, Q_l A_{lk}]+\cdots \\ &= Q_i(e^{-\lambda A})_{ik}=(e^{-\lambda\tilde{A}})_{ki}Q_i, \qquad (10) \\ S_n^{-1}P_k S_n &= P_k+\lambda A_{ki}P_i+\frac{1}{2!}\mathtt{i}\lambda^2[A_{ki}P_j, Q_l A_{lm}P_m]+\cdots \\ &= (e^{\lambda A})_{ki}P_i. \qquad (11) \end{aligned}$$

From Eq.(10) we see that when S_n acts on the n-mode coordinate eigenstate $|\vec{q}\rangle$, where $\tilde{\vec{q}}=(q_1,q_2,\cdots,q_n)$, it squeezes $|\vec{q}\rangle$ in this way:

$$S_n|\vec{q}\rangle=|\Lambda|^{1/2}|\Lambda\vec{q}\rangle,\ \Lambda=e^{-\lambda\tilde{A}},\ \ |\Lambda|\equiv\det\Lambda. \tag{12}$$

Thus S_n has the representation on the Dirac's coordinate basis $\langle\vec{q}|$[10]

$$S_n=\int d^n q S_n|\vec{q}\rangle\langle\vec{q}|=|\Lambda|^{1/2}\int d^n q\,|\Lambda\vec{q}\rangle\langle\vec{q}|,\quad S_n^\dagger=S_n^{-1}, \tag{13}$$

since $\int d^n q\,|\vec{q}\rangle\langle\vec{q}|=1$. Using the expression of $|\vec{q}\rangle$ in Fock space

$$\begin{aligned} |\vec{q}\rangle &= \pi^{-n/4}\colon \exp\left[-\tfrac{1}{2}\tilde{\vec{q}}\vec{q}+\sqrt{2}\tilde{\vec{q}}a^\dagger-\tfrac{1}{2}\tilde{a}^\dagger a^\dagger\right]|\mathbf{0}\rangle, \\ \tilde{a}^\dagger &= (a_1^\dagger,a_2^\dagger,\cdots,a_n^\dagger), \end{aligned} \tag{14}$$

and the normally ordered form of n-mode vacuum projector $|\mathbf{0}\rangle\langle\mathbf{0}|=\colon\exp[-\tilde{a}^\dagger a^\dagger]\colon$, we can put S_n into the normal ordering form,

$$\begin{aligned} S_n &= \pi^{-n/2}|\Lambda|^{1/2}\int d^n q\colon \exp[-\frac{1}{2}\tilde{\vec{q}}(1+\tilde{\Lambda}\Lambda)\vec{q}+\sqrt{2}\tilde{\vec{q}}(\tilde{\Lambda}a^\dagger+a) \\ &\quad -\frac{1}{2}(\tilde{a}a+\tilde{a}^\dagger a^\dagger)-\tilde{a}^\dagger a]\colon. \end{aligned} \tag{15}$$

To perform the integration in Eq.(15) by virtue of the IWOP technique, using the mathematical formula

$$\int d^n x\exp[-\tilde{x}Fx+\tilde{x}v]=\pi^{n/2}(\det F)^{-1/2}\exp\left[\frac{1}{4}\tilde{v}F^{-1}v\right], \tag{16}$$

then we derive

$$\begin{aligned} S_n &= \left(\frac{\det\Lambda}{\det N}\right)^{1/2} \exp\left[\frac{1}{2}\tilde{a}^\dagger\left(\Lambda N^{-1}\widetilde{\Lambda}-I\right)a^\dagger\right] \\ &\quad \times : \exp\left[\tilde{a}^\dagger\left(\Lambda N^{-1}-I\right)a\right] : \exp\left[\frac{1}{2}\tilde{a}\left(N^{-1}-I\right)a\right], \end{aligned} \tag{17}$$

where $N=(1+\widetilde{\Lambda}\Lambda)/2$. Eq.(17) is just the normal product form of S_n.

3 Squeezing property of $S_n\left|\mathbf{0}\right\rangle$

Operating S_n on the n-mode vacuum state $\left|\mathbf{0}\right\rangle$, we obtain the squeezed vacuum state

$$S_n\left|\mathbf{0}\right\rangle = \left(\frac{\det\Lambda}{\det N}\right)^{1/2} \exp\left[\frac{1}{2}\tilde{a}^\dagger\left(\Lambda N^{-1}\widetilde{\Lambda}-I\right)a^\dagger\right]\left|\mathbf{0}\right\rangle. \tag{18}$$

Now we evaluate the variances of the n-mode quadratures. The quadratures in the n-mode case are defined as

$$X_1=\frac{1}{\sqrt{2n}}\sum_{i=1}^n Q_i,\ \ X_2=\frac{1}{\sqrt{2n}}\sum_{i=1}^n P_i, \tag{19}$$

obeying $[X_1,X_2]=\frac{\mathrm{i}}{2}$. Their variances are $(\Delta X_i)^2=\left\langle X_i^2\right\rangle-\left\langle X_i\right\rangle^2$, $i=1,2$. Noting the expectation values of X_1 and X_2 in the state $S_n\left|\mathbf{0}\right\rangle$, $\left\langle X_1\right\rangle=\left\langle X_2\right\rangle=0$, then using Eqs. (10) and (11) we see that the variances are

$$\begin{aligned} (\triangle X_1)^2 &= \left\langle\mathbf{0}\right| S_n^{-1}X_1^2S_n\left|\mathbf{0}\right\rangle = \frac{1}{2n}\left\langle\mathbf{0}\right| S_n^{-1}\sum_{i=1}^n Q_i\sum_{j=1}^n Q_jS_n\left|\mathbf{0}\right\rangle \\ &= \frac{1}{2n}\left\langle\mathbf{0}\right|\sum_{i=1}^n Q_k(e^{-\lambda A})_{ki}\sum_{j=1}^n(e^{-\lambda\tilde{A}})_{jl}Q_l\left|\mathbf{0}\right\rangle \\ &= \frac{1}{2n}\sum_{i,j}^n(e^{-\lambda A})_{ki}(e^{-\lambda\tilde{A}})_{jl}\left\langle\mathbf{0}\right|Q_kQ_l\left|\mathbf{0}\right\rangle \\ &= \frac{1}{4n}\sum_{i,j}^n(e^{-\lambda A})_{ki}(e^{-\lambda\tilde{A}})_{jl}\left\langle\mathbf{0}\right|a_ka_l^\dagger\left|\mathbf{0}\right\rangle \\ &= \frac{1}{4n}\sum_{i,j}^n(e^{-\lambda A})_{ki}(e^{-\lambda\tilde{A}})_{jl}\delta_{kl} = \frac{1}{4n}\sum_{i,j}^n(\widetilde{\Lambda}\Lambda)_{ij}, \end{aligned} \tag{20}$$

similarly we have

$$(\triangle X_2)^2=\left\langle\mathbf{0}\right|S_n^{-1}X_2^2S_n\left|\mathbf{0}\right\rangle=\frac{1}{4n}\sum_{i,j}^n\left[(\widetilde{\Lambda}\Lambda)^{-1}\right]_{ij}. \tag{21}$$

Eqs. (20) -(21) are the quadrature variance formula in the transformed vacuum state acted by the operator $\exp[\mathtt{i}\lambda Q_iA_{ij}P_j]$. By observing that A in (9) is a symmetric matrix, we see

$$\sum_{i,j}^n\left[(A+\tilde{A})^l\right]_{i\,j}=2^{2l}n, \tag{22}$$

then using $A\tilde{A}=\tilde{A}A$, so $\widetilde{\Lambda}\Lambda=e^{-\lambda(A+\tilde{A})}$, a symmetric matrix, we have

$$\sum_{i,j=1}^n(\widetilde{\Lambda}\Lambda)_{i\,j}=\sum_{l=0}^\infty\frac{(-\lambda)^l}{l!}\sum_{i,j}^n\left[(A+\tilde{A})^l\right]_{i\,j}=n\sum_{l=0}^\infty\frac{(-\lambda)^l}{l!}2^{2l}=ne^{-4\lambda}, \tag{23}$$

and

$$\sum_{i,j=1}^{n}(\widetilde{\Lambda}\Lambda)^{-1}_{i\ j}=ne^{4\lambda}. \tag{24}$$

It then follows

$$\begin{aligned}(\triangle X_1)^2 &= \frac{1}{4n}\sum_{i,j}^{n}(\widetilde{\Lambda}\Lambda)_{ij}=\frac{e^{-4\lambda}}{4}, &(25)\\ (\triangle X_2)^2 &= \frac{1}{4n}\sum_{i,j}^{n}\left[(\widetilde{\Lambda}\Lambda)^{-1}\right]_{ij}=\frac{e^{4\lambda}}{4}. &(26)\end{aligned}$$

This leads to $(\triangle X_1)(\triangle X_2)=\frac{1}{4}$, which shows that S_n is a correct n-mode squeezing operator for the n-mode quadratures in Eq.(19). Furthermore, Eqs.(25) and (26) clearly indicate that the squeezed vacuum state $S_n\left|\mathbf{0}\right\rangle$ may exhibit stronger squeezing ($e^{-4\lambda}$) in one quadrature than that ($e^{-2\lambda}$) of the usual two-mode squeezed vacuum state. This is a way of enhancing squeezing.

4 The Wigner function of $S_n\left|\mathbf{0}\right\rangle$

Wigner distribution functions [12] of quantum states are widely studied in quantum statistics and quantum optics. Now we derive the expression of the Wigner function of $S_n\left|\mathbf{0}\right\rangle$. Here we take a new method to do it. Recalling that in Ref. [13] we have introduced the Weyl ordering form of single-mode Wigner operator $\Delta_1(q_1,p_1)$,

$$\Delta_1(q_1,p_1)=\vdots\,\delta(q_1-Q_1)\,\delta(p_1-P_1)\,\vdots, \tag{27}$$

its normal ordering form is

$$\Delta_1(q_1,p_1)=\frac{1}{\pi}\colon\exp\left[-(q_1-Q_1)^2-(p_1-P_1)^2\right]\colon \tag{28}$$

where the symbols $:\ :$ and $\vdots\ \vdots$ denote the normal ordering and the Weyl ordering, respectively. Note that the order of Bose operators a_1 and $a_1^\dagger$ within a normally ordered product and a Weyl ordered product can be permuted. That is to say, even though $[a_1,a_1^\dagger]=1$, we can have $:a_1a_1^\dagger:\ =\ :a_1^\dagger a_1:$ and $\vdots\,a_1a_1^\dagger\,\vdots=\vdots\,a_1^\dagger a_1\,\vdots$. The Weyl ordering has a remarkable property, i.e., the order-invariance of Weyl ordered operators under similar transformations, which means

$$U\,\vdots\,(\circ\circ\circ)\,\vdots\,U^{-1}=\vdots\,U\,(\circ\circ\circ)\,U^{-1}\,\vdots, \tag{29}$$

as if the "fence" $\vdots\ \vdots$ did not exist.

For n-mode case, the Weyl ordering form of the Wigner operator is

$$\Delta_n(\vec{q},\vec{p})=\vdots\,\delta\left(\vec{q}-\vec{Q}\right)\delta\left(\vec{p}-\vec{P}\right)\vdots, \tag{30}$$

where $\widetilde{\vec{Q}}=(Q_1,Q_2,\cdots,Q_n)$ and $\widetilde{\vec{P}}=(P_1,P_2,\cdots,P_n)$. Then according to the Weyl ordering invariance under similar transformations and Eqs.(10) and (11) we have

$$\begin{aligned}S_n^{-1}\Delta_n(\vec{q},\vec{p})\,S_n &= S_n^{-1}\,\vdots\,\delta\left(\vec{q}-\vec{Q}\right)\delta\left(\vec{p}-\vec{P}\right)\vdots\,S_n\\ &= \vdots\,\delta\left(q_k-(e^{-\lambda\tilde{A}})_{ki}Q_i\right)\delta\left(p_k-(e^{\lambda A})_{ki}P_i\right)\vdots\\ &= \vdots\,\delta\left(e^{\lambda\tilde{A}}\vec{q}-\vec{Q}\right)\delta\left(e^{-\lambda A}\vec{p}-\vec{P}\right)\vdots\\ &= \Delta\left(e^{\lambda\tilde{A}}\vec{q},e^{-\lambda A}\vec{p}\right). &(31)\end{aligned}$$

Thus using Eqs.(27) and (31) the Wigner function of $S_n |\mathbf{0}\rangle$ is

$$\begin{aligned}
& \langle \mathbf{0}| S_n^{-1} \Delta_n (\vec{q}, \vec{p}) S_n |\mathbf{0}\rangle \\
= & \frac{1}{\pi^n} \langle \mathbf{0}| : \exp[-(e^{\lambda\tilde{A}}\vec{q} - \vec{Q})^2 - (e^{-\lambda A}\vec{p} - \vec{P})^2] : |\mathbf{0}\rangle \\
= & \frac{1}{\pi^n} \exp[-(e^{\lambda\tilde{A}}\vec{q})^2 - (e^{-\lambda A}\vec{p})^2] \\
= & \frac{1}{\pi^n} \exp\left[-\tilde{\vec{q}} e^{\lambda A} e^{\lambda\tilde{A}}\vec{q} - \tilde{\vec{p}} e^{-\lambda\tilde{A}} e^{-\lambda A}\vec{p}\right] \\
= & \frac{1}{\pi^n} \exp\left[-\tilde{\vec{q}}\left(\Lambda\tilde{\Lambda}\right)^{-1}\vec{q} - \tilde{\vec{p}}\Lambda\tilde{\Lambda}\vec{p}\right],
\end{aligned} \tag{32}$$

From Eq.(32) we see that once the explicit expression of $\Lambda\tilde{\Lambda} = \exp[-\lambda(A + \tilde{A})]$ is deduced, the Wigner function of $S_n |\mathbf{0}\rangle$ can be calculated.

5 Some examples of calculating the Wigner function

For $n = 2$, form Eq.(7) we have $S_2' = \exp[\mathtt{i}2\lambda(Q_1P_2 + Q_2P_1)]$ which exhibits clearly the stronger squeezing than the usual two-mode squeezing operator S_2'. For $n = 3$, the three-mode operator [11] S_3, from Eq.(9) we see that the matrix A is $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, thus we have

$$\Lambda\tilde{\Lambda} = \begin{pmatrix} u & v & v \\ v & u & v \\ v & v & u \end{pmatrix}, \ u = \frac{2}{3}e^{2\lambda} + \frac{1}{3e^{4\lambda}}, \ v = \frac{1}{3e^{4\lambda}} - \frac{1}{3}e^{2\lambda}, \tag{33}$$

and $\left(\Lambda\tilde{\Lambda}\right)^{-1}$ is obtained by replacing λ with $-\lambda$ in $\Lambda\tilde{\Lambda}$. Thus the squeezing state $S_3 |000\rangle$ is

$$S_3 |000\rangle = A_3 \exp\left[\frac{1}{6}A_1 \sum_{i=1}^{3} a_i^{\dagger 2} - \frac{2}{3}A_2 \sum_{i<j}^{3} a_i^\dagger a_j^\dagger\right] |000\rangle, \tag{34}$$

where

$$A_1 = (1 - \mathrm{sech}2\lambda)\tanh\lambda, \ A_2 = \frac{\sinh 3\lambda}{2\cosh\lambda\cosh 2\lambda}, A_3 = \mathrm{sech}\lambda \cosh^{-1/2} 2\lambda. \tag{35}$$

In particular, for the case of the infinite squeezing $\lambda \to \infty$, Eq.(36) reduces to

$$S_3 |000\rangle \sim \exp\left\{\frac{1}{6}\left[\sum_{i=1}^{3} a_i^{\dagger 2} - 4\sum_{i<j}^{3} a_i^\dagger a_j^\dagger\right]\right\} |000\rangle \equiv |\ \rangle_{s_3}, \tag{36}$$

which is just the common eigenvector of the three compatible Jacobian operators in three-body case with zero eigenvalues [14], i.e.,

$$\begin{gathered}
(P_1 + P_2 + P_3) |\ \rangle_{s_3} = 0, \ (Q_3 - Q_2) |\ \rangle_{s_3} = 0, \\
\left(\frac{\mu_3 Q_3 + \mu_2 Q_2}{\mu_3 + \mu_2} - Q_1\right) |\ \rangle_{s_3} = 0, \ \left(\mu_i = \frac{m_i}{m_1 + m_2 + m_3}\right),
\end{gathered} \tag{37}$$

as common eigenvector

$$[P_1 + P_2 + P_3, Q_3 - Q_2] = 0, \left[\frac{\mu_3 Q_3 + \mu_2 Q_2}{\mu_3 + \mu_2} - Q_1, P_1 + P_2 + P_3\right] = 0. \tag{38}$$

Since the common eigenvector of three compatible Jacobian operators is an entangled state, the state $|\,\rangle_{s_3}$ is also an entangled state.

By using Eq.(32) the Wigner function is

$$\begin{aligned}
& \langle \mathbf{0}| S_3^{-1} \Delta_3 (\vec{q}, \vec{p}) S_3 |\mathbf{0}\rangle \\
= & \frac{1}{\pi^3} \exp\left[-\frac{2}{3} (\cosh 4\lambda + 2\cosh 2\lambda) \sum_{i=1}^{3} |\alpha_i|^2 \right] \\
& \times \exp\left\{ -\frac{1}{3} (\sinh 4\lambda - 2\sinh 2\lambda) \sum_{i=1}^{3} \alpha_i^2 \right. \\
& \left. - \frac{2}{3} \sum_{j>i=1}^{3} \left[(\cosh 4\lambda - \cosh 2\lambda) \alpha_i \alpha_j^* + (\sinh 2\lambda + \sinh 4\lambda) \alpha_i \alpha_j \right] + c.c. \right\}. \qquad (39)
\end{aligned}$$

For $n = 4$ case, the four-mode operator S_4 is

$$S_4 = \exp\{\mathtt{i}\lambda [(Q_1 + Q_3)(P_4 + P_2) + (Q_2 + Q_4)(P_1 + P_3)]\} \qquad (40)$$

the matrix $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, thus we have

$$\Lambda\tilde{\Lambda} = \begin{pmatrix} r & t & s & t \\ t & r & t & s \\ s & t & r & t \\ t & s & t & r \end{pmatrix}, \qquad (41)$$

where $r = \cosh^2 2\lambda, s = \sinh^2 2\lambda, t = -\sinh 2\lambda \cosh 2\lambda$. Then substituting Eq.(41) into Eq.(32) we obtain

$$\langle \mathbf{0}| S_4^{-1} \Delta_4 (\vec{q}, \vec{p}) S_4 |\mathbf{0}\rangle = \frac{1}{\pi^4} \exp\left\{ -2\cosh^2 2\lambda \left[\sum_{i=1}^{4} |\alpha_i|^2 + (M + M^*) \tanh^2 2\lambda + (R^* + R) \tanh 2\lambda \right] \right\}, \qquad (42)$$

where $M = \alpha_1\alpha_3^* + \alpha_2\alpha_4^*$, $R = \alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_3\alpha_4$. This form differs evidently from the Wigner function of the direct-product of usual two two-mode squeezed states' Wigner functions. In addition, using Eq. (41) we can check Eqs.(25) and (26). Further, using Eq.(41) we have

$$N^{-1} = \frac{1}{2} \begin{pmatrix} 2 & \tanh 2\lambda & 0 & \tanh 2\lambda \\ \tanh 2\lambda & 2 & \tanh 2\lambda & 0 \\ 0 & \tanh 2\lambda & 2 & \tanh 2\lambda \\ \tanh 2\lambda & 0 & \tanh 2\lambda & 2 \end{pmatrix}, \quad \det N = \cosh^2 2\lambda. \qquad (43)$$

Then substituting Eqs.(43) into Eq.(17) yields the four-mode squeezed state [11, 15],

$$S_4 |0000\rangle = \mathrm{sech} 2\lambda \exp\left[-\frac{1}{2} \left(a_1^\dagger + a_3^\dagger \right) \left(a_2^\dagger + a_4^\dagger \right) \tanh 2\lambda \right] |0000\rangle, \qquad (44)$$

from which one can see that the four-mode squeezed state is not the same as the direct product of two two-mode squeezed states in Eq.(1).

In summary, by virtue of Dirac's coordinate representation and the IWOP technique: we have shown that an n-mode squeezing operator $S_n \equiv \exp[i\lambda \sum_{i=1}^{n} (Q_i P_{i+1} + Q_{i+1} P_i))]$, $(Q_{n+1} = Q_1,\ P_{n+1} = P_1)$, is an n-mode squeezing operator which enhances the stronger squeezing for the n-mode quadratures [16]. S_n's normally ordered expansion and new n-mode squeezed vacuum states are obtained.

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